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DESIGNING NON-ROUTINE MATHEMATICAL PROBLEMS AS A CHALLENGE FOR HIGH-PERFORMING PROSPECTIVE TEACHERS

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Designing non-routine mathematical problems is a challenging task, even for excellently performing prospective teachers in primary teacher education, especially when these non-routine problems concern knowledge at the mathematical horizon (HCK). In an experimental setting these prospective teachers were challenged to design non-routine HCK problems. Interaction with peers, feedback from experts, analyzing HCK problems to find characteristics, building a repertoire of prototypes, a cyclic design process, experts who are themselves struggling in designing problems were the most important effective aspects of the learning environment that rise from this explorative study.

INTRODUCTION

All prospective teachers in the Netherlands have to pass a mathematics test in their third year in college. This test is based on notions of mathematical knowledge for teaching as formulated by Ball and colleagues (Ball, Thames, & Phelps, 2008), and mainly contains non-routine problems. Non-routine problems are problems that cannot be solved by an algorithm or other straightforward means of solution at the student teachers' disposal (Kantowski, 1977; Schoenfeld, 1985). To solve non-routine problems student teachers need to extend their mathematical knowledge in order to construct 'new' problem approaches. The third year test also contains a second type of non-routine problems. These are problems that can be solved using a standard problem approach, however doing this would be cumbersome and time-consuming. Student teachers have to construct efficient approaches to solve these problems and to finish the test within the given time.

See for instance the following problem:

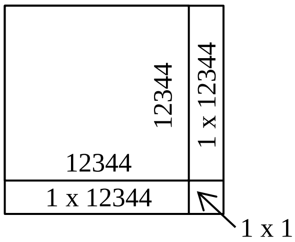
Compare 12344 x 12344 with 12345 x 12345.

The difference between these products is…

- *a. 2 x 12344*
- *b. 2 x 12344 + 1*
- *c. 2 x 12345*
- *d. 2 x 12345 + 1*

To solve this problem some student teachers will choose the standard problem approach and subtract the product of 12344 x 12344 from the product of 12345 x 12345. It takes a lot of time to do this and thereafter they have to compare their result with the several answer options, which is also time-consuming.

Student teachers who recognize 12345 x 12345 as $(12344 + 1)^2 = 12344^2 + 2 \times 12344 \times 1 + 1^2$ can easily find option b as the right answer, but most of our student teachers don't have this knowledge available and need to construct a 'new' problem approach, for instance drawing a diagram.



This is an elegant and efficient problem approach, but you need to be creative to construct such a strategy yourself. For most student teachers solving non-routine mathematical problems is quite challenging. Practice is needed to prepare for the test. However, test-specific practice materials are hardly available. To fill this gap, in the college years 2013-2014 and 2014-2015 groups of excellently performing prospective primary teachers employed their excellence in mathematics in designing mathematical problems for their peers who were preparing for the mathematics test.

Designing problems at this high level is difficult and not a requirement for becoming a primary teacher. Even so, the excellent student teachers started this task with much enthusiasm, because it challenged them on their own high level. Moreover they wanted to provide their peers with suitable training materials. And finally, designing problems requires that one takes the perspective of the problem solver and wonder what he or she could do or think, and which different problem approaches he could choose. Doing so will enrich the designer's mathematical knowledge and it will provide a deeper mathematical understanding, which in its turn supports better mathematics teaching.

Both groups of excellent student teachers analyzed, evaluated and designed non-routine problems concerning all the three types of mathematical subject matter knowledge as distinguished by Ball, Thames and Phelps (2008), namely Common Content Knowledge (CCK), that is the subject-specific knowledge needed to recognize and solve mathematics problems in day-to-day-life, Specialized Content Knowledge (SCK), the – for a teacher – professional mathematical knowledge to understand, assess and evaluate the mathematical productions of students, and Horizon Content Knowledge (HCK), the knowledge that exceeds the mathematics of the school type the professional is teaching. HCK comprises knowledge of how mathematical topics are related over the span of mathematics included in the curriculum. All these three types of knowledge are represented in the mathematical test. The first group of excellently performing prospective teachers did finally learn to design non-routine CCK and SCK problems, but at the end of the year they were unable to design non-routine problems that needed HCK (Kool & Keijzer, 2015). The expert problem designers who guided the group used their expertise in focusing the 2014-2015 group on designing CCK, SCK **and** HCK problems.

BACKGROUND

Students solving non-routine mathematical problems need to construct new problem approaches. This involves that once having solved the problem this specific type of problem can become a routine problem. Learning mathematics means that you will always encounter new non-routine problems. Students have to learn to construct new problem approaches. Knowledge, skills, metacognition and self-confidence are valuable for this process and the use of heuristics can be useful too (Verschaffel, De Corte, Lasure, Van Vaerenbergh, Bogaerts, & Ratinckx, 1999).

Heuristics are general advice, search rules, rules of thumb and informal approaches that might help problem solvers when solving non-routine mathematical problems. Drawing a diagram or model, looking for patterns, making suppositions, working backwards, and simplifying a problem are examples of heuristics. Heuristics do not guarantee that a problem will be solved, but they offer a chance to find a solution that is sought for. A problem solver can use heuristics to explore and analyze non-routine problems (Verschaffel, et al., 1999).

Solving non-routine problems is challenging, but designing these problems is even more demanding. Excellent students like to reach for demanding goals like this one, but in working on such a challenge they have specific needs. On the one hand they need sufficient subject specific input, they want to be a member of a learning community, they feel responsible for tasks and want to decide how cooperatively formulated goals could and should be achieved (Van Tassel-Baska, 1993; Borasi, Fonzi, Smith, & Rose, 1999; Swan, Holmes, Vargas, Jennings, Meier, & Rubenfeld, 2002; Heller, Perleth, & Lim, 2005; Feldhusen, 2005; Subotnik & Jarvin, 2005; Scager, 2013). On the other hand, they need 'scaffolds' from experts who show good examples, provide feedback, evaluate and appreciate their work, and support them in performing tasks they cannot (currently) perform independently (Bain, 2004; Van Geert & Steenbeek, 2005; Brixler, 2007; Frey & Fisher, 2010). This support is needed but should fade over time, because not doing so will have a negative impact on the excellent performers and influence their motivation to participate in the task (Keller, 2010).

RESEARCH QUESTION

The 2013-2014 group learned to design non-routine problems for CCK and SCK, but failed in designing non-routine HCK problems. Therefore, there is a need to develop a learning environment, including goals, roles, peers, teachers, sources, meetings and appointments, in a course setting for one year for the 2014-2015 group of high-performing prospective teachers, that would support them in designing non-routine HCK problems (besides CCK and SCK problems). As a consequence the research question is:

What are characteristics of a learning environment for high-performing prospective teachers that supports them in designing non-routine HCK problems?

METHOD

Design research is an appropriate research approach for the given research question (Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006). The design of the experimental learning environment is described in Table 1, where interventions are connected to student teachers' hypothetical learning trajectory, HLT (Simon, 1995). The interventions are based on reflection on and evaluations of experiences in the 2013-2014 group (Kool & Keijzer, 2015).

interventions in the learning environment	hypothetical learning trajectory (HLT)
prospective teachers analyze and evaluate examples of non-routine HCK problems using a provisional characteristics list based on the perspective of the problem solver	prospective teachers become familiar with these non-routine problems; they learn about the mathematics involved and what these problems require from the problem solver
prospective teachers are asked to look for non-routine HCK problems using the characteristics list; these problems are discussed	prospective teachers learn to choose the perspective of the problem solver and use this to recognize non-routine HCK problems
prospective teachers analyze and evaluate non-routine HCK problems and improve and extend the characteristics list in terms of problem solving characteristics (heuristics) and they collect a list of prototypes of HCK problems	prospective teachers become familiar with the character of non-routine HCK problems and heuristics used in solving these, they can recognize the quality of the problems and they build a repertoire of prototypes of HCK problems
prospective teachers participate in master classes from an experienced designer of non-routine problems, focusing on the cyclic process of designing non-routine problems (production – evaluation – improvement); prospective teachers see the expert sometimes struggling designing the problems.	prospective teachers reflect on the cyclic process of designing non-routine problems, try to follow the same process in designing non-routine HCK problems and feel supported in their struggle in designing non-routine HCK problems.

Table 1: Interventions and HLT for learning to design non-routine HCK problems

During seven meetings with the group one of the experts led the session, while the other took field notes. These notes were analyzed to see whether and how the prospective teachers reacted to interventions and whether they developed as was predicted in the HLT. In addition, research data was collected through:

- analyzing problems the prospective teachers designed during the year,
- asking prospective teachers to reconstruct and describe the (cyclic) design process of some of their problems,
- a structured group interview with the prospective teachers about aspects of the learning environment they find stimulating or frustrating.

RESULTS

During the year, eight high-performing student teachers participated in the project. After some time they succeeded in designing non-routine problems about daily life experiences (CCK) and teaching experiences (SCK). The student teachers did not spontaneously design HCK problems and when they were stimulated to do so they only produced a few routine HCK problems instead of non-routine problems. Their explanation was that they did not have enough experience with this kind of problems, which play a smaller part in the teacher education curriculum. It turned out that they could solve these problems, but most of the time only in one way. They experienced HCK problems as uncommon and unfamiliar. Their repertoire of examples and problem approaches concerning HCK was poor, and not flexible enough to design this kind of non-routine problems.

Characteristics of non-routine HCK problems

To make student teachers more familiar with non-routine HCK problems the experts provided many suitable examples. They stimulated student teachers to solve these problems in different ways and to analyze the character of the problems. They asked the student teachers to change their perspective to that of the problem solver and imagine which heuristics he could use. In doing this the group constructed the following criteria for non-routine HCK problems. These problems:

- challenge the student to reason, backtrack, abstract, generalize, declare, explain, prove and justify, to look for the best problem approach and ask himself why this is the best one,
- challenge the student to discover mathematical structures (rules, patterns),
- require that the student understand thoroughly the knowledge, skills and problem approaches he is using.

The high-performing student teachers who constructed these criteria concluded that they overlap partially, and that a good HCK problem does not have to meet all the criteria. This self-constructed list helped them to recognize HCK problems, to find HCK problems in textbooks and study materials and to evaluate them.

They discussed for instance the question if and why the following problem was a non-routine HCK problem:

Given: John calculates the greatest common divisor of two numbers. He multiplies both numbers with the number a. Question: What do you know about the greatest common divisor of the two products?

Using their criteria the student teachers decided that this was a good non-routine HCK problem, because knowing the algorithm to find the greatest common divisor of two numbers is not enough to solve this problem. One has to understand the concept of the greatest common divisor. Some reasoning is required in solving the problem. Of course a conjecture can be checked filling in several numbers, but this will not provide proof for all sets of two numbers.

A collection of prototypes of non-routine HCK problems

After some time the student teachers became more familiar with non-routine HCK problems. They had solved, analyzed and evaluated many examples, and they improved and used their list of characteristics. In spite of that, designing non-routine HCK problems was still difficult for them. After some time they were able to make variations of existing problems and they decided that it could be helpful to collect prototypes of HCK problems. These included problems that:

- ask for why or why not. *Why* $6 : \frac{1}{2} = 6 \times 2$*?*
- present a proposition and ask under what conditions the proposition is correct. *When does the product of two primes have exactly 3 divisors?*
- ask if and why a certain proposition is always correct. *If you add three consecutive numbers your sum will always be odd. Is this true?*

- ask to predict the effect of a given mistake, like mixing up the digits in a number, or interchanging two operations. *I want to multiply two fractions. I only multiply the numerators and forget to multiply the denominators. What will be the effect on the result?*
- give the result of a calculation and ask what the calculation has been. *What number do you need to multiply 14 to get 18 as a result?*
- ask to apply mathematical knowledge in an expanded situation. *You know the sum of the angles of a triangle, what will be the sum of the angles of a hexagon?*

Although the student teachers made some variations on the prototypes, they were still struggling with the task and gave up quite fast. One of the experts showed the student teachers that even experts can struggle with this. Student teachers learned that it is almost impossible to design a perfect non-routine HCK problem out of the blue. Designing a problem starts with an idea, a basic problem, a mathematical concept. A cyclic process of evaluating and improving your idea a number of times is necessary to achieve the final product.

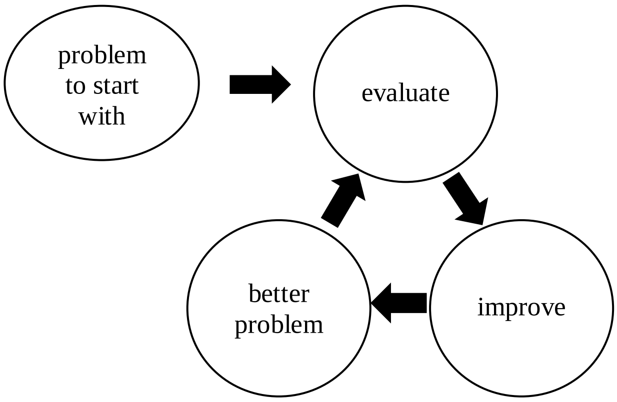


Figure 1: The cyclic process of designing and improving non-routine mathematical problems

In the master classes the expert generally started designing by using an existing problem, sometimes just a routine problem and wondered aloud: ‘What can I do to change this into a non-routine problem?’ He showed how he could find his inspiration in the above list of prototypes and spoke explicitly about what was going on in his mind, like: ‘Perhaps I can replace numbers by letters (variables), turn the problem upside down by giving the answer and asking for the question, or withhold some information.’ Each of his designed problems was evaluated by the student teachers using the list of criteria, wondering what the problem solver has to do to solve the problem: Is this a non-routine HCK problem, is it an interesting one or can we improve it? The student teachers experienced that the first attempts of the expert were not always perfect. He needed some rounds of evaluating and improving to design a better problem and sometimes he really had to struggle to finally be successful. The following impression of a master class illustrates this.

Like always at the start of his master class the expert asked the student teachers to choose a mathematical topic they wanted some support on in designing problems. This time they chose repeating decimal fractions. They were familiar with fractions that equals standard decimals like 0,333… and 0,111… They tried to make non-routine variations on these examples and designed problems like finding the fractions that equals 0,555… and 0,777… but were not satisfied with their productions because they were still routine and too easy. Indeed the problem solver can see at a glance that these numbers were multiples of 0,111…. These problems were unsuitable to prepare for the test and they wonder if the expert could design a suitable variation on these problems. The expert could not immediately satisfy their desire. To start with he asked them to find a fraction that equals 0,123123123… The student teachers struggled with this problem and were not very successful in solving it until one of them demonstrated a problem approach he had learned once: If a = 0,123123123… then 1000a = 123,123123123… and therefor 999a = 123. So a = $\frac{123}{999} = \frac{41}{333}$. Afterwards the student teachers evaluated that this was an interesting problem but that it was not suitable to prepare for the test, because if a problem solver knows this problem approach this problem has become a routine problem but if he doesn't know this problem approach it is too hard and perhaps impossible to construct a suitable problem approach himself. The expert was invited to find a better variation. He tried the following one: 'Find a fraction that equals 0,1333…' After working on this problem the student teachers could appreciate this example because they discovered and constructed a new problem approach: 0,1333… = 0,333… – 0,2 = $\frac{1}{3} - \frac{1}{5} = \frac{2}{15}$. At the same time they regretted that the standard problem approach was still usable in solving this case. One of the student teachers suggested an adjustment of the problem: 'Write 0,1333 as an subtraction of two fractions.' This was a good attempt, but unfortunately it was still possible for the problem solver to use the standard problem approach to find $\frac{2}{15}$ as a result and after that he had countless possibilities to construct subtractions ending on $\frac{2}{15}$. Finally the expert designed: 'Write 0,13333 as a subtraction of two fractions with numerator 1.' This final adjustment caused that the problem met the requirements and after that it inspired the student teachers to design many variations like: 'Write 0,2666… as an addition of two fractions with numerator 2' and 'Write 0,291666… as a subtraction of two fractions with the property that the denominator of the first fraction equals the numerator of the second'. Each designed problem was evaluated by the student teachers and they were quite satisfied about the problem structure they finally constructed together with the expert. For instance the last example was appreciated because it cannot be solved by the standard problem approach, the specific conditions make the problem challenging, but the average student teacher should have enough mathematical knowledge to construct his own problem approach.

For most of the student teachers it was an eye opener and an encouragement to see that even an expert in designing problems needs time, support and perseverance to design a good problem finally.

A cyclic designing process

The experts observed the development of the student teachers and analyzed their designs during the whole year. After the master classes they discovered that the student teachers were more likely to vary on one of the prototypes to design non-routine HCK problems, that they used the criteria list to discuss and evaluate their productions and that they tried harder to improve their first attempts. In other words the student teachers tried to follow the cycle of designing, evaluating and improving problems and produced several stages of a problem before the final stage was reached.

For instance student teacher Lieke developed the following HCK problem in stages. She started with this design:

The price of a pair of shoes was increased by 20%. At the end of the season the price was decreased by 20%. What do you know of the new price compared with the original one?

A. *The new price is higher than the original price.*
B. *The new price is lower than the original price.*
C. *The new price equals the original price.*

The student teachers discussed whether this was a HCK problem. First they thought this was a day-to-day-life problem (CCK) but in the end they realized that although the situation could be realistic it was in the first place a problem that asked you to reason about a mathematical phenomenon in itself. But they were not convinced that the problem solver really has to reason to solve this problem. If you pick a price for the shoes, you can calculate and find the answer. Some student teachers remarked that this problem is part of their basic knowledge. They recognized the problem and knew the answer. So they decided that this was a routine HCK problem. Lieke used the feedback to improve her problem:

Sale in the shoe shop: all prices are lowered by 20%! The shop assistant makes a mistake, she decreases the price of my new shoes by 20 euro, and that means that she gives me 5 euro too much discount. What was the original price of my new shoes?

The group of student teachers agreed that this was a real non-routine problem. Without using equations and variables, one cannot use a straightforward approach to solve this problem. One needs to backtrack and in constructing the problem approach the problem solver has to thoroughly understand calculation with percentages. This was a suitable non-routine HCK problem for student teachers. Lieke was happy with this judgment, and this experience motivated her to design a variation that was even more challenging.

A shoe shop had a very special way of giving discount. If you buy two pairs of shoes, for the cheapest pair you will get this discount: subtract 20 euros and thereafter decrease the price by 20%. For the most expensive pair they first decrease the price by 20% and thereafter subtract 20 euros. A customer buys two pairs of shoes and she ends up paying the same price for both pairs. What was the difference in euros between the original prices of the two pairs of shoes?

The student teachers recognized the quality of this problem and decided that this variation requires mathematical reasoning at a quite high level. The development of Lieke's designing work shows how interaction, feedback and evaluating problems could stimulate varying problems and improving them. The student teachers recognized this and wanted feedback on all their designed problems, from peers and experts. They experienced that it is useful to give your first idea a chance to grow. This did not guarantee success, but the approach of throwing away each initially rejected idea and starting with a new one was far less successful. In the end half of the student teachers were able to design non-routine HCK problems. They still found it difficult, but several times they could bring their struggle to a good end.

At the end of the year the student teachers evaluated the trajectory they followed in a group interview to find out which aspects of the learning environment were valuable in reaching the final result. During the group interview the student teachers were asked to mention aspects of the learning environment that were frustrating and other aspects that were supportive and stimulating. The following aspects were mentioned by at least two of the eight student teachers.

During their learning-trajectory student teachers found it frustrating that the experts let them experience:

- that they have not encountered enough non-routine HCK problems during their study, that their HCK repertoire was too small,
- that although they could solve HCK problems, their HCK was not rich and deep enough to design HCK problems.

Student teachers found it stimulating and supportive to discover:

- that designing non-routine HCK problems is a cyclic process of repeated evaluating and improving a starting problem or first idea,
- that a list of characteristics of non-routine HCK problems can be used to evaluate problems from the perspective of the problem solver and improve these,
- that a list of prototypes of HCK problems can give inspiration to start the cyclic process of designing problems,
- that feedback from experts can help to go on in the cyclic process of designing problems
- that interaction and discussion with peers can be valuable too,
- that even experts struggle when designing non-routine HCK problems.

CONCLUSION AND DISCUSSION

Finally all the student teachers developed their mathematical knowledge and their designing capacities; as said before, only half of the participants reached the final goal and could design non-routine HCK problems independently. Student teachers with a less flexible and rich HCK and HCK problem repertoire still needed a lot of support to do this, but even these student teachers admitted that they found it valuable for their future profession as a teacher to have solved, analyzed, evaluated so many HCK problems. After the trajectory they seemed able to construct HCK problems with some expert help. Moreover, during this trajectory their HCK had grown.

High performing prospective teachers who want to learn to design non-routine HCK problems in the first place will need scaffolds and feedback to achieve this challenging goal. The learning environment must also provide support and input from expert designers, and target both characteristics of non-routine HCK problems and a repertoire of HCK problem types. Following a cyclic process of designing problems is helpful, as is experiencing that designing non-routine HCK problems is hard even for expert designers. These features of the learning environment were not effective for all of the excellent student teachers. This raises new questions concerning characteristics of the participants and the learning environment.

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